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Quantum Tunneling from Apparent Horizon of Rainbow-FRW Universe

Kai Lin · ShuZheng Yang

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Abstract The quantum tunneling from the apparent horizon of rainbow-FRW universe is studied in this paper. We apply the semi-classical approximation, which is put forward by Parikh and Wilczek et al., to research on the scalar field particles tunneling from the apparent horizon of the rainbow-FRW universe, and then use the spin 1/2 Fermions tunneling theory, which brought forward by Kerner and Mann firstly, to research on the Fermions Hawking radiation via semi-classical approximation. Finally, we discuss the meanings of the quantum effect via Finsler geometry.

Keywords Rainbow-FRW universe · Hawking radiation · Apparent horizon · Quantum tunneling

1 Introduction

Popularly, researchers think that Hawking radiation is quantum effect near the horizon of black holes [1, 2], and there are two viewpoints which could explain the effect. A point of view believes it comes from the quantum vacuum fluctuation near the horizon of the black holes, so the positive-negative particle pairs could come out, and the negative energy particle may cross the horizon due to the negative energy orbit inside black holes, while the positive energy particle should eradiate to infinite finally and becomes the Hawking radiation. Via the point of view, people have put forward several methods, such as anomalies theory, to research on the Hawking radiation [3–6]. On the other hand, the Hawking radiation could be viewed as the realized particles radiation which becomes from virtual particles inside black holes across the horizon, and Parikh and Wilczek et al. have brought forward the semi-classical approximation to research on the tunneling radiation via the point of view [7–15]. In the semi-classical approximation method, the WKB approximation is used, so

K. Lin (✉) · S.Z. Yang
Institute of Theoretical Physics, China West Normal University, Nanchong 637002, China
e-mail: lk314159@126.com

S.Z. Yang
e-mail: szyangcwnu@126.com

that the tunneling rate $\Gamma \propto \exp(-2\,\mathrm{Im}\, S)$ is applied (S is the first term from the expanding classical action via $\hbar$), and then, considering the relation between tunneling rate and Hawking temperature $\Gamma \propto \exp(-\omega/T)$, the Hawking temperature of the horizon can be obtained. It is a valid method which researches on the Hawking radiation from horizon of black holes. In 2007, Ryan Kerner and R.B. Mann put forward a method to study the Fermions tunneling radiation from black holes [16, 17], then Li, Chen, Jiang, Zeng et al. have developed and generalized the method in several space-times [18–26], so that the method matures gradually.

Recently, Cai et al. have discussed the Hawking quantum tunneling radiation from the apparent horizon of FRW universe [27]. In their paper, the radial coordination $\tilde{r} = a(t)r$ is defined, and the semi-classical approximation of Parikh and Wilczek et al. is used. Their work suggests that the quantum tunneling is the nature near the horizon of both black holes and universe.

In this paper, we generalize Cai's work in the research about the 0 spin scalar particles and 1/2 spin Fermions tunneling in Rainbow-FRW universe, and obtain significative results about tunneling rate and Hawking temperature finally. The remainders of the paper are outlined as follows; In Sect. 2, the Rainbow-FRW metric is introduced briefly, and the scalar particles tunneling and Fermions tunneling near the horizon are studied in Sects. 3 and 4. Finally, some conclusions and discussions about the relation between the Hawking radiation nature of the apparent horizon and Finsler geometry are included in Sect. 5.

2 Gravity's Rainbow Theory and Rainbow-FRW Metric

In 1910's, Einstein studied the first relativistic cosmology model, it is a symbol as outset of cosmology. From then on, the physics cosmology is established after nearly a century of research. In cosmology, the Inflationary universe theory and Big Bang theory are proved by several cosmology observations. Although, recently, there are some difficulties, such as dark energy and dark matter, most of researchers believe that the theories are reasonable. In the cosmological scale, people think that the 3-dimensional space is symmetrical [28]. According to the principle, researchers put forward a famous Friedman-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{1}$$

On the other hand, in the quantum gravitation theory, the gravity's rainbow metric theory was proposed. In the theory, the Einstein's theory is modified, and the energy momentum invariant is modified as [29, 30]

$$l_1^2 E^2 - l_2^2 \vec{p}\cdot\vec{p} = m^2 \tag{2}$$

where, we let $c = \hbar = 1$, l_1 and, l_2 are two correction terms which correlate with probe particles' energy. Of course, in lower energy theory, the effect of the correction terms will be ignored, while the evident effect of them could be observed in high energy physics experiment. Generalizing the theory in gravity theory, the Einstein field equation is modified as

$$G_{\mu\upsilon}(E) = 8\pi G(E)T_{\mu\upsilon}(E) + g_{\mu\upsilon}\Lambda(E) \tag{3}$$

and the Rainbow-FRW metric is written as [29]

$$ds^2 = -\frac{dt^2}{l_1^2} + \frac{a^2(t)}{l_2^2}\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{4}$$

Generalizing the ideal of Cai in the space-time, we can define

$$\tilde{r} = l_1 a(t) r / l_2 \tag{5}$$

In the coordination, the nature which is observed by observers depends on the energy of probe particles as well as the coordination functions. On the other hand, according to the astronomical observation recently, our universe should be flat, so we just research on the flat universe, and the k in (4) is 0. Therefore, the Rainbow-FRW metric could be written as

$$ds^2 = -(1-\tilde{r}^2H^2)\frac{dt^2}{l_1^2} + \frac{d\tilde{r}^2}{l_1^2} - 2H\tilde{r}\frac{dtd\tilde{r}}{l_1^2} + \frac{\tilde{r}^2}{l_1^2}(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6}$$

where, $H = \dot{a}/a$ is Hubble factor. Due to the apparent horizon equation $g^{\mu\upsilon}\partial_\mu\tilde{r}\partial_\upsilon\tilde{r} = 0$, we can obtain the apparent horizon of space-time (6) is $\tilde{r}_A = 1/H$. In fact, the apparent horizon is no other than the Hubble horizon of the metric. Using the metric, we will discuss the quantum tunneling from the apparent horizon of Rainbow-FRW universe.

3 Scalar Field Particles Tunneling of Rainbow-FRW Universe

In curved space-time, the Klein-Gordon equation, which can describe the nature of scalar field particles, is

$$-\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\upsilon}\sqrt{-g}\frac{\partial\varphi}{\partial x^\upsilon}\right) + \frac{m^2}{\hbar^2}\varphi = 0 \tag{7}$$

Applying the semi-classical approximation, the function could be written as

$$\varphi = Ce^{\frac{i}{\hbar}S(t,\tilde{r},\theta,\varphi)} \tag{8}$$

and the little contribution of terms which have $\hbar$ as multiplicator in the space-time (6) is ignored, so that Hamilton-Jacobi equation is

$$-l_1^2\left(\frac{\partial S}{\partial t}\right)^2 - 2\tilde{r}Hl_1^2\frac{\partial S}{\partial\tilde{r}}\frac{\partial S}{\partial t} + l_1^2(1-\tilde{r}^2H^2)\left(\frac{\partial S}{\partial\tilde{r}}\right)^2 + \frac{l_1^2}{\tilde{r}^2}\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{l_1^2}{\tilde{r}^2\sin^2\theta}\left(\frac{\partial S}{\partial\varphi}\right)^2 + m^2 = 0 \tag{9}$$

Decomposing the variables, we can get

$$S = -\omega t + R(\tilde{r}) + Y(\theta,\varphi) \tag{10}$$

Equation (10) could be decomposed as radial equation and non-radial equation

$$-\omega^2 + 2\tilde{r}H\omega\frac{\partial R}{\partial\tilde{r}} + (1-\tilde{r}H)(1+\tilde{r}H)\left(\frac{\partial R}{\partial\tilde{r}}\right)^2 + \frac{m^2}{l_1^2} = \frac{\lambda}{l_1^2\tilde{r}^2} \tag{11}$$

$$l_1^2\left(\frac{\partial Y}{\partial\theta}\right)^2 + \frac{l_1^2}{\sin^2\theta}\left(\frac{\partial Y}{\partial\varphi}\right)^2 + \lambda = 0 \tag{12}$$

where, λ is constant. In the case that (12) is true, from (11), we can obtain

$$\frac{dR(\tilde{r})}{d\tilde{r}} = \frac{-\tilde{r}H \pm \sqrt{\tilde{r}^2H^2 - (1-\tilde{r}^2H^2)(m^2 l_1^{-2}\omega^{-2} - \lambda\tilde{r}^{-2}l_1^{-2}\omega^{-2} - 1)}}{(1-\tilde{r}H)(1+\tilde{r}H)}\omega \tag{13}$$

Near the apparent horizon, the result of (13) is

$$\operatorname{Im} S = \operatorname{Im} R = \operatorname{Im} R_+(\tilde{r}) - \operatorname{Im} R_-(\tilde{r}) = \pi\tilde{r}_A\omega \tag{14}$$

where, Im stands for the imaginary part, while R_+ is the part of out-coming solution, and R_- is the part of in-going solution, so the tunneling rate and Hawking temperature are

$$\Gamma = \exp(-2\mathrm{Im}R) = \exp(-2\pi\tilde{r}_A\omega) \tag{15}$$

$$T_A = \frac{1}{2\pi\tilde{r}_A} \tag{16}$$

The results show that there is Hawking radiation near the apparent horizon, and Hawking radiation is not only the nature of black holes' horizon but also the universe's apparent horizon.

4 Spin 1/2 Fermions Tunneling of the Apparent Horizon of Rainbow-FRW

As we all know, the Dirac equation in curved space-time is

$$\boldsymbol{\gamma}^\mu D_\mu\boldsymbol{\Psi} + \frac{m}{\hbar}\boldsymbol{\Psi} = 0 \qquad \mu = t, \tilde{r}, \theta, \varphi \tag{17}$$

where

$$D_\mu = \partial_\mu + \frac{i}{2}\Gamma^{\alpha\beta}_{\mu}\Pi_{\alpha\beta} \tag{18}$$

$$\Pi_{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta] \tag{19}$$

The gamma matrix must satisfy

$$\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\upsilon\} = 2g^{\mu\upsilon}I \tag{20}$$

In the space-time (6), the gamma matrix could be written as

$$\gamma^t = l_1\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{21}$$

$$\gamma^{\tilde{r}} = \tilde{r}Hl_1\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + l_1\begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix} \tag{22}$$

$$\gamma^\theta = \frac{l_1}{\tilde{r}}\begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix} \tag{23}$$

$$\gamma^{\varphi} = \frac{l_1}{\tilde{r}\sin\theta}\begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix} \tag{24}$$

where

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{25}$$

$$\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{26}$$

$$\sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{27}$$

Now, let's decompose the $\mathbf{\Psi}$ as the spin up case and spin down case

$$\mathbf{\Psi}_{\uparrow} = \begin{bmatrix} A \\ 0 \\ B \\ 0 \end{bmatrix} e^{\frac{i}{\hbar}S_{\uparrow}} \tag{28}$$

$$\mathbf{\Psi}_{\downarrow} = \begin{bmatrix} 0 \\ C \\ 0 \\ D \end{bmatrix} e^{\frac{i}{\hbar}S_{\downarrow}} \tag{29}$$

First of all, we research on the spin up case. Using the semi-classical approximation, we ignore the little effect of the terms which have $\hbar$ as multiplicator, and the equations can be obtained

$$\left(il_1\frac{\partial S_{\uparrow}}{\partial t} + i\tilde{r}Hl_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} - im\right)A + l_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}}B = 0 \tag{30}$$

$$-\frac{Bl_1}{\tilde{r}}\left(\frac{\partial S_{\uparrow}}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial S_{\uparrow}}{\partial \varphi}\right) = 0 \tag{31}$$

$$l_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}}A - \left(il_1\frac{\partial S_{\uparrow}}{\partial t} + i\tilde{r}Hl_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} + im\right)B = 0 \tag{32}$$

$$-\frac{Al_1}{\tilde{r}}\left(\frac{\partial S_{\uparrow}}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial S_{\uparrow}}{\partial \phi}\right) = 0 \tag{33}$$

In fact, (31) and (33) describe the nature of angular terms, and (30) and (32) describe the nature of radial terms. However, because Hawking radiation is radial nature of space-time, we just pay attention to (30) and (33)

$$\begin{pmatrix} il_1\frac{\partial S_{\uparrow}}{\partial t} + i\tilde{r}Hl_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} - im & l_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} \\ l_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} & -il_1\frac{\partial S_{\uparrow}}{\partial t} - i\tilde{r}Hl_1\frac{\partial S_{\uparrow}}{\partial \tilde{r}} - im \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{34}$$

It could be easily seen that if and only if the determinant of the coefficient matrix vanishes, does the equation have a non-trivial solution for A, and B, so we have

$$\begin{vmatrix} il_1\frac{\partial S_\uparrow}{\partial t}+i\tilde{r}Hl_1\frac{\partial S_\uparrow}{\partial \tilde{r}}-im & l_1\frac{\partial S_\uparrow}{\partial \tilde{r}} \\ l_1\frac{\partial S_\uparrow}{\partial \tilde{r}} & -il_1\frac{\partial S_\uparrow}{\partial t}-i\tilde{r}Hl_1\frac{\partial S_\uparrow}{\partial \tilde{r}}-im \end{vmatrix}=0 \tag{35}$$

namely

$$-\omega^2+2\tilde{r}H\omega\frac{\partial S_\uparrow}{\partial \tilde{r}}+(1-\tilde{r}H)(1+\tilde{r}H)\left(\frac{\partial S_\uparrow}{\partial \tilde{r}}\right)^2+\frac{m^2}{l_1^2}=0 \tag{36}$$

Via (36) we can obtain

$$\frac{dS_\uparrow(\tilde{r})}{d\tilde{r}}=\frac{-\tilde{r}H\pm\sqrt{\tilde{r}^2H^2-(1-\tilde{r}^2H^2)(m^2l_1^{-2}\omega^{-2}-1)}}{(1-\tilde{r}H)(1+\tilde{r}H)}\omega \tag{37}$$

Near the apparent horizon, the result of (34) is

$$\operatorname{Im} S_\uparrow=\operatorname{Im} S_{\uparrow+}(\tilde{r})-\operatorname{Im} S_{\uparrow-}(\tilde{r})=\pi\tilde{r}_A\omega \tag{38}$$

where Im stands for the imaginary part, while $S_{\uparrow+}$ is the part of out-coming solution, and $S_{\uparrow-}$ is the part of in-going solution, so the tunneling rate and Hawking temperature are

$$\Gamma=\exp(-2\operatorname{Im} S_\uparrow)=\exp(-2\pi\tilde{r}_A\omega) \tag{39}$$

$$T_A=\frac{1}{2\pi\tilde{r}_A} \tag{40}$$

Similarly, we can prove that the results of spin down case and spin up are equivalent. Therefore, both tunneling rate and Hawking temperature of scalar field particles tunneling and Fermions tunneling are the same.

5 Conclusion

According to our work, the observers who stand in space-time (6) fail to find the difference via probe particles with different energy. However, from (5), we can see that the coordination of observers depends on the energy of probe particles, and they could find the energy dependent of Hawking temperature and tunneling rate if they stand in other coordination. Therefore, quantum tunneling from apparent horizon of Rainbow-FRW universe depends on the energy of probe particles at all events.

In math, the energy of probe particle can be described by tangent vector, so Rainbow-FRW metric is not a Riemannian geometry metric, but a Finsler geometry metric. According to modern differential geometry theory, the Finsler geometry is geometry beyond quadratic metric's bound, so the form of correction terms in gravity's rainbow theory fails to be obtained via classical Einstein field equation which bases on the Riemannian geometry theory, while it could be gotten in a new equation which bases on the Finsler geometry. In fact, Girelli F. et al. had put forward that the Rainbow metric is Finsler metric in 2007, and researched some elementary problems of the gravity's rainbow theory via Finsler theory [31].

The conclusion in this paper suggests that, in the observations of high energy universe particle, observers could find that the Hawking radiation near the apparent horizon of universe is very different from the results from FWR metric, so the research about Finsler gravity theory could be very important.

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